and bulk deformation; l, displacement; A_ε, B_ε, kinetic parameters of deformation; σ', Stefan–Boltzmann coefficient; $\varkappa$, empirical coefficient; h, τ, difference-grid steps over the coordinate and time. Indices: 0, initial; e, external gas flow; S, surface; ch, chemical entrainment; I, condensed phase (body); II, gas phase in material; s, layer number; Σ, total; BD, m, ED, beginning, maximum, and end of decomposition; BP, EP, beginning and end of plastic state.

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VARIATIONAL ESTIMATE OF THE EFFECTIVE GENERALIZED CONDUCTIVITY TENSOR OF A TWO-PHASE MEDIUM WITH AN ANISOTROPIC DISTRIBUTION OF PHASES

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An inequality is found for the effective generalized conductivity tensor of a two-phase medium with an anisotropic distribution of phases.

There are a large number of calculations of the effective generalized conductivity of a two-phase inhomogeneous medium; see for example [1-3]. The idea of a generalized conductivity derives from a local coupling of two vectors fields (denoted by **E** and **j**) by a linear relation with the proportionality factor dependent on the material characteristics. In the absence of sources, one of the fields will be potential, and the other solenoidal, and the equations for the spatial distribution of fields will be given by

$$\operatorname{rot}\mathbf{E} = 0;\ \operatorname{div}\mathbf{j} = 0;\ \mathbf{j} = \Lambda\mathbf{E}, \tag{1}$$

where the generalized conductivity will in general be a tensor of the second rank. In the present paper we consider the case of a scalar $\Lambda > 0$ which is more often encountered in practice.

The set of equations (1) describes processes of heat conduction, diffusion, electrical conduction and also the electric field distribution in a dielectric and the magnetic field in a material with the magnetic permeability differing from unity. In the particular case of heat conduction, **E** is the temperature gradient and **j** is the heat flux; then Λ will be the thermal conductivity.

Various methods have been used to calculate the effective conductivity. The variational method has been used in only a relatively few cases, as can be seen in the reviews [1-3]. However, variational methods have several advantages which show considerable promise. For example we show that the variational inequality obtained here yields not only an approximate effective generalized conductivity but also allows calculation of

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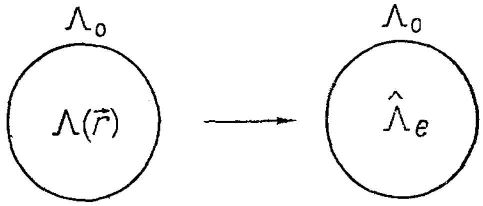

Fig. 1. Model used in calculation of the effective generalized conductivity tensor.

the error. The advantage of the variational method is that the inequality can be improved at any step of the calculation by choosing more exact trial functions; this in turn is done by using more complete information on the spatial distribution of the inhomogeneities.

The variational method does have a difficulty in that some of its results are not explicitly interpretable physically. For example, it is difficult to give a physical interpretation to the quantity Λ_0 introduced in [4, 5] in a variational calculation based on a plane-wave expansion of the fields. Also in [6, 7] where a singular approximation was studied, an analogous parameter was called the generalized conductivity of the reference medium, which does not clarify very much.

The treatment given here allows an extension of the results of [4] to an anisotropic spatial distribution of phases, and also gives a simple physical interpretation of the parameter Λ_0 used in [4-7]. Unlike [4, 5], we donot uase a plane wave expansion of the fields but base our treatment on the model developed in [8-10].

In [8] it was proposed to calculate the effective parameters for a medium with isotropic spherical inclusions of another phase statistically distributed in it, by considering a sphere of inhomogeneous material placed in a medium with material constants equal to those of the matrix. Changing the form of this model somewhat, we consider an inhomogeneous sphere in a medium with a generalized conductivity Λ_0 (see Fig. 1) assuming that the radius of the sphere is much larger than the characteristic size of the inhomogeneities. Then according to the definition of the effective generalized conductivity tensor, the average field characteristics in the medium will be unchanged if we replace the inhomogeneous sphere by a homogeneous one with generalized conductivity tensor $\hat{\Lambda}_e$. One of these average characteristics will be the polarizability tensor $\hat{\alpha}$ from the electrostatics of dielectrics [11]. It couples the dipole moment p of the sphere placed in a uniform field $\mathbf{E}_0$ with the vector $\mathbf{E}_0$ by the relation

$$\mathbf{p} = \hat{\alpha} \cdot \mathbf{E}_0, \tag{2}$$

where the dot between tensors of different rank means a contraction over the pair of inner indices. In particular, the polarizability tensor of the uniform sphere shown in Fig. 1 is given by

$$\hat{\alpha} = R^3 \left(\frac{1}{\Lambda_0} \hat{\Lambda}_e - \hat{e} \right) \cdot \left(\frac{1}{\Lambda_0} \hat{\Lambda}_e + 2\hat{e} \right)^{-1}. \tag{3}$$

The calculation of $\hat{\alpha}$ is done using the set of equations (1). Two approaches are possible, using either the scalar potential defined by $\mathbf{E} = \mathbf{E}_0 - \nabla\varphi$ or the vector potential defined by $\mathbf{j} = \Lambda_0(\operatorname{rot} \mathbf{A} + \mathbf{E}_0)$. In [9] these two approaches were studied by variational methods and the following inequalities were obtained for the polarizability tensor of an inhomogeneous sphere:

$$2\pi \mathbf{E}_0 \cdot \hat{\alpha} \cdot \mathbf{E}_0 \leqslant \frac{1}{2} \left(\left\langle \frac{\Lambda}{\Lambda_0} \right\rangle - 1 \right) \mathbf{E}_0^2 V + W(\varphi); \tag{4}$$

$$2\pi \mathbf{E}_0 \cdot \hat{\alpha} \cdot \mathbf{E}_0 \geqslant \frac{1}{2} \left(\left\langle \frac{\Lambda_0}{\Lambda} \right\rangle - 1 \right) \mathbf{E}_0^2 V + W(\mathbf{A}),$$

$$W(\varphi) = \int_\Omega \left[\frac{1}{2} \frac{\Lambda}{\Lambda_0} (\nabla\varphi)^2 - \left(\frac{\Lambda}{\Lambda_0} - 1 \right) \nabla\varphi \cdot \mathbf{E}_0 \right] dV; \tag{5}$$

$$W(\mathbf{A}) = \int_\Omega \left[\left(1 - \frac{\Lambda_0}{\Lambda} \right) \mathbf{E}_0 \cdot \operatorname{rot} \mathbf{A} - \frac{\Lambda_0}{2\Lambda} (\operatorname{rot} \mathbf{A})^2 \right] dV. \tag{6}$$

The integration in (5) and (6) goes over all space Ω. The scalar potential on the right-hand side of (5) can be represented as an integral

$$\varphi(\mathbf{r}) = \int_V \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' \tag{7}$$

over the volume of the sphere. The vector potential on the right-hand side of (6) can be represented in similar form

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mathbf{P}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' \tag{8}$$

where the vector function is integrated over the volume of the sphere. It is natural here to interprete $\mathbf{P}(\mathbf{r}')$ as the polarization vector inside the sphere, i.e., as the density of dipole moment sources of fields $\mathbf{E}$ and $\mathbf{j}$. As shown in [9], the inequality (4) is converted into an equality for the actual polarization distribution inside the sphere. In the variational method, we consider $\mathbf{P}(\mathbf{r}')$ in (7) and (8) to the some trial functions.

Let the lower $\hat{\alpha}_i$ and upper $\hat{\alpha}_s$ bounds for the polarizability tensor of the sphere occur for values of Λ_0 which we call Λ_{0i} and Λ_{0s}. Then from the inequalities

$$\hat{\alpha}_i < R^3(\hat{\Lambda}_e - \Lambda_{0i}\hat{e}) \cdot (\hat{\Lambda}_e + 2\Lambda_{0i}\hat{e})^{-1};$$
$$\hat{\alpha}_s > R^3(\hat{\Lambda}_e - \Lambda_{0s}\hat{e}) \cdot (\hat{\Lambda}_e + 2\Lambda_{0s}\hat{e})^{-1},$$

obtained from (3), there follows an inequality for the effective generalized conductivity tensor

$$\Lambda_{0i}\left(\hat{e} + \frac{2}{R^3}\hat{\alpha}_i\right) \cdot \left(\hat{e} - \frac{1}{R^3}\hat{\alpha}_i\right)^{-1} < \hat{\Lambda}_e < \Lambda_{0s}\left(\hat{e} + \frac{2}{R^3}\hat{\alpha}_s\right) \cdot \left(\hat{e} - \frac{1}{R^3}\hat{\alpha}_s\right)^{-1}, \tag{9}$$

where a tensor inequality $\hat{a} > \hat{b}$, as understood from matrix theory [12], is equivalent to the statement that the tensor $\hat{a} - \hat{b}$ is positive definite.

In order to calculate α_s and α_i, we choose as test functions for φ and $\mathbf{A}$ superpositions of the fields corresponding to uniformly polarized regions V_1 and V_2 occupied by the two different phases. In order to be definite we take the generalized conductivity of the second phase to be larger than that of the first, $\Lambda_2 > \Lambda_1$. In the discussion below we will use extensively the idea of the depolarization tensor of an arbitrary region of space, first introduced in [10]. For example, in region V_1 the depolarization tensor is given by either of the two following equivalent formulas

$$4\pi V_1 \hat{N}_1 = \int_{V_1} \nabla \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV dV' = \int \frac{d\mathbf{S} d\mathbf{S}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{10}$$

In the first equality the integration is carried out twice over the volume V_1, while in the second equality, it is carried out twice over the surface bounding V_1.

We consider the properties of the depolarization tensor which follow from the definition (10). Since the trace of the tensor under the first integral in (10) is equal to $4\pi\delta(\mathbf{r} - \mathbf{r}')$ it is obvious that the trace of $\hat{N}_1$ is equal to unity. It can also be seen from (10) that $\hat{N}_1$ is symmetric, and the product $\mathbf{P}_1 \cdot \hat{N}_1 \cdot \mathbf{P}_1$ differs from the electrostatic energy of a uniformly polarized region V_1 only by a positive constant factor; this shows explicitly that $\hat{N}_1$ is positive definite. Hence $\hat{N}_1$ has all of the properties of a depolarization tensor as defined in the usual way for an ellipsoid [11], therefore (10) can be considered as a generalization of the concept of depolarization tensor to a region of space of arbitrary shape. For a space with the symmetry of a cube, $\hat{N}_1 = {}^1/_3\hat{e}$; this value also results when we average $\hat{N}_1$ over a statistically isotropic distribution of phase 1.

The functionals (5) and (6) are calculated with the help of the trial functions given above and the following equations

$$\langle \nabla\varphi_1 \rangle = \frac{4\pi}{3} f_1 \mathbf{P}_1; \quad \langle \nabla\varphi_1 \rangle_{V_1} = 4\pi \hat{N}_1 \cdot \mathbf{P}_1; \tag{11}$$

$$\langle \operatorname{rot} \mathbf{A}_1 \rangle = \frac{8\pi}{3} f_1 \mathbf{P}_1; \quad \langle \operatorname{rot} \mathbf{A}_1 \rangle_{V_1} = 4\pi(\hat{e} - \hat{N}_1) \cdot \mathbf{P}_1, \tag{12}$$

where φ_1 and $\mathbf{A}_1$ are determined by the integrals (7) and (8) and correspond to the potentials of a uniformly polarized region V_1. Potentials φ_2 and A_2 will satisfy identical relations, where φ_2 and A_2 are potentials created by a uniformly polarized region occupied by phase 2. They can be obtained from (11) and (12) by simply replacing subscript 1 and 2.

Functional (5) can be expressed in the form

$$W(\varphi)=\sum_{i=1}^{2}\left\langle 2\pi\mathbf{P}_i\cdot\nabla\varphi+(\lambda_i-1)\left(\frac{1}{2}\nabla\varphi-\mathbf{E}_0\right)\cdot\nabla\varphi\right\rangle_{V_i}V_i. \tag{13}$$

Suppose $\lambda_i=\Lambda_i/\Lambda_0<1$, for this case it is sufficient to choose $\Lambda_0>\Lambda_2$. Then the first inequality of (4) is weakened if in place of $W(\varphi)$ on the right-hand side we put

$$W^*(\varphi)=W(\varphi)+\frac{1}{2}\sum_{i=1}^{2}(1-\lambda_i)\langle(\nabla\varphi-\langle\nabla\varphi\rangle_{V_i})^2\rangle_{V_i}V_i,$$

which is larger than $W(\varphi)$ because $\lambda_i<0$. From (13) we have

$$W^*(\varphi)=\sum_{i=1}^{2}\langle\nabla\varphi\rangle_{V_i}\left[2\pi\mathbf{P}_i+(1-\lambda_i)\left(\mathbf{E}_0-\frac{1}{2}\langle\nabla\varphi\rangle_{V_i}\right)\right]V_i, \tag{14}$$

and from (11) we find

$$\begin{aligned}\frac{1}{4\pi}\langle\nabla\varphi\rangle_{V_i}&=\hat{N}_1\cdot\mathbf{P}_1+\left(\frac{1}{3}\hat{e}-\hat{N}_1\right)\cdot\mathbf{P}_2;\\ \frac{1}{4\pi}\langle\nabla\varphi\rangle_{V_2}&=\hat{N}_2\cdot\mathbf{P}_2+\left(\frac{1}{3}\hat{e}-\hat{N}_2\right)\cdot\mathbf{P}_1,\end{aligned} \tag{15}$$

so that $W^*(\varphi)$ can be represented as a quadratic form in the polarization vectors $\mathbf{P}_i$ of the phases. The best upper bound $\hat{\alpha}_s$ will be obtained when $W^*(\varphi)$ reaches a minimum:

$$2\pi\mathbf{E}_0\cdot\hat{\alpha}_s\cdot\mathbf{E}_0=\frac{1}{2}\langle\lambda-1\rangle\mathbf{E}_0^2V+\min_{\mathbf{P}_i}W^*(\varphi). \tag{16}$$

In calculating $\hat{\alpha}_i$ it is convenient to use a similar representation for $W(\mathbf{A})$. In this case one must choose Λ_0 less than Λ_1 so that now $\lambda_i>1$. The formulas corresponding to (14), (15), and (16) will be

$$W^*(\mathbf{A})=\sum_{i=1}^{2}\langle\operatorname{rot}\mathbf{A}\rangle_{V_i}\cdot\left[\left(1-\frac{1}{\lambda_i}\right)\left(\mathbf{E}_0+\frac{1}{2}\langle\operatorname{rot}\mathbf{A}\rangle_{V_i}\right)-2\pi\mathbf{P}_i\right]V_i; \tag{17}$$

$$\begin{aligned}\frac{1}{4\pi}\langle\operatorname{rot}\mathbf{A}\rangle_{V_1}&=(\hat{e}-\hat{N}_1)\cdot\mathbf{P}_1-\left(\frac{1}{3}\hat{e}-\hat{N}_1\right)\cdot\mathbf{P}_2;\\ \frac{1}{4\pi}\langle\operatorname{rot}\mathbf{A}\rangle_{V_2}&=-\left(\frac{1}{3}\hat{e}-\hat{N}_2\right)\cdot\mathbf{P}_1+(\hat{e}-\hat{N}_2)\cdot\mathbf{P}_2;\end{aligned} \tag{18}$$

$$2\pi\mathbf{E}_0\cdot\hat{\alpha}_i\cdot\mathbf{E}_0=\frac{1}{2}\left\langle 1-\frac{1}{\lambda}\right\rangle\mathbf{E}_0^2V+\max_{\mathbf{P}_i}W^*(\mathbf{A}). \tag{19}$$

Solving for $\hat{\alpha}_s$ from (14) through (16) (for $\Lambda_0>\Lambda_2$) and solving for $\hat{\alpha}_i$ from (17) through (19) (for $\Lambda_0<\Lambda_1$) and then substituting the resulting values into inequality (9), we find the following inequality for $\hat{\Lambda}_e$:

$$\hat{\Lambda}_i(\Lambda_0)|_{\Lambda_0\leqslant\Lambda_1}<\hat{\Lambda}_e<\Lambda_s(\Lambda_0)|_{\Lambda_0\geqslant\Lambda_2}. \tag{20}$$

In the special case when $\Lambda_0=\Lambda_2$,

$$\hat{\alpha}_s=\frac{V_1}{4\pi}\left(\frac{\Lambda_2}{\Lambda_1-\Lambda_2}\hat{e}+\hat{N}_1\right)^{-1}, \tag{21}$$

and when $\Lambda_0=\Lambda_1$,

$$\hat{\alpha}_i = \frac{V_2}{4\pi}\left(\frac{\Lambda_1}{\Lambda_2-\Lambda_1}\hat{e}+\hat{N}_2\right)^{-1}. \tag{22}$$

In this case (20) takes the form

$$\Lambda_1\left\{\hat{e}+f_2\left[\left(\frac{\Lambda_1}{\Lambda_2-\Lambda_1}-\frac{f_2}{3}\right)\hat{e}+\hat{N}_2\right]^{-1}\right\}\leqslant\hat{\Lambda}_e\leqslant\Lambda_2\left\{\hat{e}+f_1\left[\left(\frac{\Lambda_2}{\Lambda_1-\Lambda_2}-\frac{f_1}{3}\right)\hat{e}+\hat{N}_1\right]^{-1}\right\}. \tag{23}$$

For the effective generalized conductivity tensor of a two-phase medium in two dimensions* we can repeat nearly exactly the above discussion for the three-dimensional case, finally obtaining the two-dimensional analogs of (21) through (23):

$$\hat{\alpha}_s = \frac{V_1}{2\pi}\left(\frac{\Lambda_2}{\Lambda_1-\Lambda_2}\hat{e}+\hat{N}_1\right)^{-1}; \tag{24}$$

$$\hat{\alpha}_i = \frac{V_2}{2\pi}\left(\frac{\Lambda_1}{\Lambda_2-\Lambda_1}\hat{e}+\hat{N}_2\right)^{-1}; \tag{25}$$

$$\Lambda_1\left\{\hat{e}+f_2\left[\left(\frac{\Lambda_1}{\Lambda_2-\Lambda_1}-\frac{f_2}{2}\right)\hat{e}+\hat{N}_2\right]^{-1}\right\}\leqslant\hat{\Lambda}_e\leqslant\Lambda_2\left\{\hat{e}+f_1\left[\left(\frac{\Lambda_2}{\Lambda_1-\Lambda_2}-\frac{f_1}{2}\right)\hat{e}+\hat{N}_1\right]^{-1}\right\}, \tag{26}$$

where V_1 and V_2 are two-dimensional volumes (areas) and f_1 and f_2 are the volume fractions occupied by phases 1 and 2. The depolarization tensors $\hat{N}_1$ and $\hat{N}_2$ are positive-definite second rank tensors, symmetric and with unit trace. They can be represented by 2×2 matrices and calculated using formulas analogous to (10). For example, for $\hat{N}_1$ we have

$$2\pi V_1\hat{N}_1 = \int_{V_1}\nabla\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\,dVdV' = -\int(\ln|\mathbf{r}-\mathbf{r}'|)\,d\mathbf{S}d\mathbf{S}', \tag{27}$$

where as in (10) the integration in the first integral goes twice over V_1 and in the second twice over the surface bounding V_1; in the latter integral $d\mathbf{S}$ is a vector line element pointing in the direction of the outward normal to the surface.

Note that the depolarization tensors, which are characteristic of an anisotropic distribution of phases, are introduced in the solution of the problem in a natural way. When the phases have isotropic distributions we have $\hat{N}_1 = \hat{N}_2 = {}^1/_3\hat{e}$ in three dimensions and $\hat{N}_1 = \hat{N}_2 = {}^1/_2\hat{e}$ in two dimensions, and (23) and (26) reduce to

$$\Lambda_1\left[1-3f_2\left(f_2+\frac{2\Lambda_1+\Lambda_2}{\Lambda_1-\Lambda_2}\right)^{-1}\right]\leqslant\Lambda_e\leqslant\Lambda_2\left[1-3f_1\left(f_1+\frac{2\Lambda_2+\Lambda_1}{\Lambda_2-\Lambda_1}\right)^{-1}\right]; \tag{28}$$

$$\Lambda_1\left[1-2f_2\left(f_2+\frac{\Lambda_1+\Lambda_2}{\Lambda_1-\Lambda_2}\right)^{-1}\right]\leqslant\Lambda_e\leqslant\Lambda_2\left[1-2f_1\left(f_1+\frac{\Lambda_1+\Lambda_2}{\Lambda_2-\Lambda_1}\right)^{-1}\right]. \tag{29}$$

Inequality (28) is the same as obtained in [4] by other methods. Hence (23) generalizes the results of [4] to the case where the distribution of phases is anisotropic. The parameter Λ_0 as used by us can be given an explicit physical interpretation; it is the generalized conductivity of a medium into which is placed an inhomogeneous sphere. Equations (21) and (22) can be used to estimate the polarizability of particles of complex shape. These estimates are important in problems involving the scattering of electromagnetic waves from small dielectric particles [10], however we do not discuss them in detail here. Our results unify problems which at first glance appear to be very different by providing a single approach to their solution.

The basic difficulty in applying inequalities (23) and (26) in practice is the complexity of calculating the depolarization tensor $\hat{N}_1$. The depolarization tensor $\hat{N}_2$ can be expressed in terms of $\hat{N}_1$ using the relation

$$f_1(\hat{e}-3\hat{N}_1) = f_2(\hat{e}-3\hat{N}_2) \tag{30}$$

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*This case occurs in calculations of effective parameters in thin films, for example.

in three dimensions and

$$f_1(\hat{e}-2\hat{N}_1)=f_2(\hat{e}-2\hat{N}_2) \tag{31}$$

in two dimensions. The general properties of the depolarization tensor have been discussed above. We add here that one can often use the symmetry of the region V_1 to determine the direction of the principal axis of $\hat{N}_1$ without actually doing the integrals (10) or (27). For example if V_1 has a symmetry axis higher than second order, this axis will be one of the principal axes and the other two mutually perpendicular principal axes can be chosen arbitrarily in the plane perpendicular to the symmetry axis. For a circular cylinder of finite length, the principal value of the depolarization tensor corresponding to the axis along the cylinder can be found from the relation

$$N_1=\frac{1}{y}[\Phi(0)-\Phi(y)], \tag{32}$$

where

$$\Phi(y)=\frac{4}{3\pi}(1+y^2)^{1/2}[y^2K(k)+(1-y^2)E(k)]-y.$$

Tables of the complete elliptic integrals K(k) and E(k) are given in [13]. The other two principal values are $N_2=N_3=\frac{1}{2}(1-N_1)$.

The principal axes of the depolarization tensor for a rectangle will be parallel to the sides of the rectangle. The principal values are

$$N_b=\frac{2}{\pi}\operatorname{arctg}z+\frac{1}{2\pi}\left[z\ln\left(1+\frac{1}{z^2}\right)-\frac{1}{z}\ln(1+z^2)\right]$$

and $N_a=1-N_b$, where N_b, N_a are the principal values in the b and a directions, respectively.

The above expressions for the depolarization tensors of a cylinder and rectangle can be used directly to estimate the effective generalized conductivity tensor of a medium with a small concentration of parallel identical finite circular cylinders or rectangles (in the two-dimensional case). A simple calculation shows that at low concentrations the depolarization tensor of a region occupied by inclusions differs from the depolarization tensor of an isolated inclusion by a quantity proportional to the volume concentration, for example f_1 in (23) and (26). One then expands these inequalities in powers of f_1 and keeps the linear term in f_1. $\hat{N}_1$ is replaced by the depolarization tensor of a cylinder for three dimensions or by that for a rectangle in two dimensions.

NOTATION

$\mathbf{E}$, potential vector field; $\mathbf{j}$, solenoid vector field; Λ, generalized conductivity; Λ_0, generalized conductivity of a medium containing a nonuniform sphere; $\mathbf{p}$, total dipole moment of sources inside the sphere; $\hat{\alpha}$, polarizability tensor of the sphere; $\mathbf{E}_0$, uniform potential field in the medium in the absence of the sphere; $\hat{\Lambda}_e$, effective generalized conductivity tensor; R, radius of the sphere; $\hat{e}$, unit tensor; ∇, gradient operator; φ, the potential of $\mathbf{E}$; $\mathbf{A}$, the vector potential of $\mathbf{j}$; $\langle\ \rangle$, operation of averaging over the volume of the sphere; V, volume of the sphere; Ω, all space; $\mathbf{r}$, radius vector; $\mathbf{P}$, polarization; $\hat{\alpha}_i$ and $\hat{\alpha}_s$, lower and upper bounds of the polarizability tensor of the sphere; Λ_{0i} and Λ_{0s}, values of Λ_0 corresponding to $\hat{\alpha}_i$ and $\hat{\alpha}_s$; V_i, $f_i=V_i/V$, Λ_i, P_i, $\hat{N}_i$, volume, volume fraction, generalized conductivity, polarizability, and depolarization tensor of the i-th phase; $i=1, 2$; $\langle\ \rangle_{V_i}$, operation of averaging over the volume V_i, $\delta(\mathbf{r})$, Dirac delta function; $\lambda_i=\Lambda_i/\Lambda_0$; $\lambda=\Lambda/\Lambda_0$; N_1, N_2, N_3, principal values of the depolarization tensor; y, ratio of the half-length of the cylinder to its radius; K(k) and E(k), complete elliptic integrals with modulus $k=(1+y^2)^{-1/2}$; N_a and N_b, principal values of the depolarization tensor for a rectangle corresponding to principal axes directed along sides a and b, $z=a/b$.

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VARIATIONAL METHOD OF DETERMINING THE HYDRODYNAMIC PARAMETERS IN CONVECTIVE HEAT-TRANSFER PROBLEMS FOR SEPARATION FLOWS IN CHANNELS

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A method is given for computing viscous fluid flows in a channel by using a variational formulation.

The application of iteration methods [1-6] to compute the convective heat transfer in a separation flow in a channel requires substantial expenditures of machine time, which is associated mainly with the slow convergence of the iteration process for the hydrodynamic equations.

Utilization of direct methods, including variational, results in a significant reduction in the computation time, as a rule, although it also complicates the algorithm for the solution.

A certain hybrid scheme is proposed in this paper that combines finite-difference and variational-difference computation schemes, which turn out to be relatively simply in realization on an electronic computer while at the same time sufficiently economical in the sense of the computation time.

The scheme is developed in application to specific cases of the flow in cylindrical or plane channels behind a sudden expansion and is based on an explicit method for solving all equation in the longitudinal (cruising) coordinate x and an implicit method in the transverse coordinate y.

A feature of the method is that the solution is sought in the form $u = \overline{u} + \delta u$; $v = \overline{v} + \delta v$, where $\overline{u}$ is the first approximation obtained from (1) by the factorization method, $\overline{v}$ is determined from the continuity equation (3), δu, δv are the refining corrections obtained from the condition of minimum work of the hydrodynamic forces on a finite set of closed contours (closedness of the contour permits elimination of the pressure from a number of unknowns).

The method mentioned permits obtaining a "good" solution more rapidly for the problem under consideration than in [1-5], say, because of the abrupt reduction in the number of iterations typical for variational methods. At the same time, such a combined approach is simpler, and (in this case) more economical than the ap-

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